

STOCHASTIC CONTROL ON THE HALF-LINE AND APPLICATIONS TO THE OPTIMAL DIVIDEND/CONSUMPTION PROBLEM

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ABSTRACT. We consider a stochastic control problem with the assumption that the system is controlled until the state process breaks the fixed barrier. Assuming some general conditions, it is proved that the resulting Hamilton Jacobi Bellman equations has smooth solution. The aforementioned result is used to solve the optimal dividend and consumption problem. In the proof we use a fixed point type argument, with an operator which is based on the stochastic representation for a linear equation.

1. INTRODUCTION

Our main motivation is to prove general existence theorem for the classical solution $(\mathcal{C}^{2,1}((0, +\infty) \times [0, T)) \cap \mathcal{C}([0, +\infty) \times [0, T]))$ to the parabolic HJB equation of the form

$$D_t + \frac{1}{2}\sigma^2(x, t)D_x^2 u + \max_{\delta \in D} (i(x, t, \delta)D_x u + h(x, t, \delta)u + f(x, t, \delta)) = 0, \quad (x, t) \in (0, +\infty) \times [0, T)$$

with the boundary condition $u(x, t) = \beta(x, t)$, for $(x, t) \in \partial((0, +\infty) \times [0, T))$. The set $D \subset \mathbb{R}^l$ is assumed to be compact. Such equation appears naturally in finite time stochastic control problems where the system is stopped when the controlled process hits the barrier. In this paper we would like to put the emphasis on problems connected to dividend optimization and consumption-investment problems. We treat the aforementioned HJB equations as semilinear equation and prove our main result under more general setting. We believe that this will open the gate to consider many singular and ergodic problems. Due to our knowledge such problem has never been solved under such general setting.

Our work is structured as follows. In Section 2 we consider general semilinear equation and prove our main theorem using a fixed point approach. In Section 3 we present the application of the result to stochastic control problems including stochastic control for dividend problems. Section 4 is dedicated to applications our main result to some unrestricted consumption-investment problems, which were introduced, in some specific examples, by Korn and Kraft [2], Kraft and Steffensen [3].

Key words and phrases. Cauchy-Dirichlet problem, Hamilton Jacobi Bellman equation, optimal dividend problem, uncertain time horizon, optimal consumption-investment problem.

2. GENERAL RESULTS

Our main objective here is to prove the existence result for a smooth solution to the equation

$$(2.1) \quad \begin{cases} D_t + \frac{1}{2}\sigma^2(x,t)D_x^2u + H(D_xu, u, x, t) = 0, & (x, t) \in (0, +\infty) \times [0, T), \\ u(x, t) = \beta(x, t), & (x, t) \in \partial((0, +\infty) \times [0, T)). \end{cases}$$

We find it helpful to associate equation (2.1) with the one dimensional diffusion given by

$$dX_t = \sigma(X_t, t)dW_t$$

where $(W_t, t \geq 0)$ is a one dimensional Brownian motion. The symbol $\mathbb{E}_{x,t}f(X_s)$ is used to denote the expected value when the system starts at time t from the state $x > 0$. For a notational convenience we sometimes use $\mathbb{E}f(X_s)$ as well. Let $\tau(x, t)$ denote now the first time the process hits 0 i.e.

$$\tau(x, t) := \inf\{k \geq 0 \mid X_k(x, t) = 0\}.$$

We make the following assumptions.

Assumption 1.

A1) The coefficient $\sigma > \varepsilon > 0$ is uniformly bounded, Lipschitz continuous on compact subsets in $[0, +\infty) \times [0, T]$, and Lipschitz continuous in x uniformly with respect to t .

A2) The function β is bounded and Lipschitz continuous.

A3) The Hamiltonian H is Hölder continuous on compact subsets of $\mathbb{R}^2 \times [0, +\infty) \times [0, T]$. Moreover, let there exists $K > 0$ such that for all $(p, u, x, t), (\bar{p}, \bar{u}, x, t) \in \mathbb{R}^2 \times [0, +\infty) \times [0, T]$

$$(2.2) \quad \begin{aligned} |H(p, u, x, t)| &\leq K(1 + |u| + |p|), \\ |H(p, u, x, t) - H(\bar{p}, \bar{u}, x, t)| &\leq K(|u - \bar{u}| + |p - \bar{p}|). \end{aligned}$$

A4) There exists a constant $L > 0$ that for all $(x, t), (\bar{x}, t) \in [0, +\infty) \times [0, T]$

$$(2.3) \quad |\mathbb{E}\tau(x, t) \wedge T - \mathbb{E}\tau(\bar{x}, t) \wedge T| \leq L|x - \bar{x}|.$$

Let $C_b^{1,0}$ stands for the space of all functions that are continuous, bounded and have the first derivative with respect to x , which is also continuous and bounded. The space is equipped with the family of norms:

$$\|u\|_\kappa := \sup_{(x,t) \in [0, +\infty) \times [0, T]} e^{-\kappa(T-t)} |u(x, t)| + \sup_{(x,t) \in (0, +\infty) \times [0, T]} e^{-\kappa(T-t)} |D_x u(x, t)|.$$

Note that the space $C_b^{1,0}$ together with $\|\cdot\|_\kappa$ forms a Banach space.

Additionally, there is also a need to use the space $C_b^{0,0}$ consisting of all functions which are continuous and bounded. This space is considered together with the norm

$$\|u\|_\kappa^0 = \sup_{(x,t) \in [0,+\infty) \times [0,T]} e^{-\kappa(T-t)} |u(x,t)|.$$

We consider first the linear equation

$$(2.4) \quad \begin{cases} D_t + \frac{1}{2}\sigma^2(x,t)D_x^2 u + f(x,t) = 0, & (x,t) \in (0,+\infty) \times [0,T], \\ u(x,t) = 0, & (x,t) \in \partial((0,+\infty) \times [0,T]). \end{cases}$$

Proposition 2.1. *Suppose that condition A1 is satisfied and let the function f be Hölder continuous on compact subsets of $[0,+\infty) \times [0,T]$ then there is a smooth $(\mathcal{C}^{2,1}((0,+\infty) \times [0,T]) \cap \mathcal{C}([0,+\infty) \times [0,T]))$ solution to (2.4).*

Proof. If the function f is Lipschitz continuous in x and Hölder continuous in t on compact subsets of $[0,+\infty) \times [0,T]$, then the claim was proved by Rubio [4, Theorem 3.1]. To extend it to weaker condition we take the sequence of mollifiers $(\varphi_n, n \in \mathbb{N})$ and approximate the function f by the sequence $f_n = f \star \varphi_n$ and use (E10) from Fleming and Rishel [5] to prove uniform bounds for Hölder norms of the corresponding sequence of smooth solutions. The standard application of Arzela-Ascoli Lemma ends the proof. \square

The first step in our reasoning is to prove estimates for $\|u\|_\kappa$ where u is a solution to (2.4).

Proposition 2.2. *Let $u_f \in \mathcal{C}^{2,1}((0,+\infty) \times [0,T]) \cap \mathcal{C}([0,+\infty) \times [0,T])$ denote the classical solution to (2.4) and assume that conditions A1 and A4 are satisfied. Then, there exists a constant $M > 0$ such that for all functions f , being bounded and Hölder continuous on compact subsets of $(0,+\infty) \times [0,T]$, we have*

$$\|u_f\|_\kappa \leq \frac{M}{\kappa} \|f\|_\kappa^0.$$

Proof. Due to the Feynman - Kac representation we have

$$u_f(x,t) = \mathbb{E}_{x,t} \int_t^{\tau(x,t) \wedge T} f(X_s, s) ds.$$

Hence,

$$\begin{aligned} e^{-\kappa(T-t)} u_f(x,t) &= \mathbb{E}_{x,t} e^{-\kappa(T-t)} \int_t^{\tau(x,t) \wedge T} e^{\kappa(T-s)} e^{-\kappa(T-s)} f(X_s, s) ds \\ &\leq \|f\|_\kappa^0 e^{-\kappa(T-t)} \int_t^T e^{\kappa(T-s)} ds = \frac{1}{\kappa} \|f\|_\kappa^0 e^{-\kappa(T-t)} (e^{\kappa(T-t)} - 1) \leq \frac{1}{\kappa} \|f\|_\kappa^0. \end{aligned}$$

The derivative $D_x u_f$ is estimated using the Lipschitz constant. Fix $x, \bar{x} \in [0,+\infty]$ and assume that $x > \bar{x}$. In particular, this assumption implies that $\tau(x,t) > \tau(\bar{x},t)$.

We have

$$|u(x, t) - u(\bar{x}, t)| \leq \left| \mathbb{E} \int_t^{\tau(x, t) \wedge T} f(X_s(x, t)) ds - \mathbb{E} \int_t^{\tau(x, t) \wedge T} f(X_s(\bar{x}, t)) ds \right| \\ + \left| \mathbb{E} \int_t^{\tau(x, t) \wedge T} f(X_s(\bar{x}, t)) - \int_t^{\tau(\bar{x}, t) \wedge T} f(X_s(\bar{x}, t)) \right| =: I_1 + I_2$$

The first integral can be estimated using the theory of fundamental solutions for parabolic equations. The fundamental solution is denoted by $\Gamma(x, t, z, s)$. Recall that there exist $c, C > 0$ such that

$$|\Gamma(x, t, y, s)| \leq \frac{C}{(s-t)^{-1/2}} \exp\left(-c \frac{|y-x|^2}{(s-t)}\right), \\ |D_x \Gamma(x, t, y, s)| \leq \frac{C}{(s-t)} \exp\left(-c \frac{|y-x|^2}{(s-t)}\right)$$

(see Friedman [6, Chapter 1, Theorem 11]).

We have

$$I_1 = \left| \mathbb{E} \int_t^{\tau(x, t) \wedge T} f(X_s(x, t), s) ds \right| \leq \int_t^T |\mathbb{E} \chi_{\{\tau(x, t) > s\}} [f(X_s(x, t), s) - f(X_s(\bar{x}, t), s)]| ds.$$

If $\tau(x, t) \leq s$, then $\tau(\bar{x}, t) \leq s$ and consequently $f(X_s(x, t), s) = f(X_s(\bar{x}, t), s)$. Therefore,

$$I_1 \leq \int_t^T |\mathbb{E} [f(X_s(x, t), s) - f(X_s(\bar{x}, t), s)]| ds \leq \int_t^T \int_{\mathbb{R}} |f(z, s)| |\Gamma(x, t, z, s) - \Gamma(\bar{x}, t, z, s)| dz ds \\ = |x - \bar{x}| \int_t^T \int_{\mathbb{R}} |f(z, s)| |\Gamma(x^*, t, z, s)| dz ds = |x - \bar{x}| \int_t^T \int_{\mathbb{R}} |f(z, s)| \frac{C}{(s-t)} \exp\left(-c \frac{|z-x^*|^2}{s-t}\right) dz ds \\ = |x - \bar{x}| \int_t^T \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}} |f(z, s)| \frac{C}{\sqrt{s-t}} \exp\left(-c \frac{|z-x^*|^2}{s-t}\right) dz ds \\ \leq \frac{C\sqrt{2\pi}}{\sqrt{2c}} \|f\|_{\kappa}^0 |x - \bar{x}| \int_t^T \frac{1}{\sqrt{t-s}} e^{\kappa(T-s)} ds.$$

Thus, by multiplying both sides by $e^{-\kappa(T-t)}$, we obtain

$$e^{-\kappa(T-t)} I_1 \leq \frac{C\sqrt{2\pi}}{\sqrt{2c}} \|f\|_{\kappa}^0 |x - \bar{x}| \int_t^T \frac{1}{\sqrt{t-s}} e^{\kappa(t-s)} ds \leq \frac{C\sqrt{2\pi}}{\sqrt{2c}} \|f\|_{\kappa}^0 |x - \bar{x}| \int_t^T \frac{1}{\sqrt{t-s}} ds.$$

For the second integral we have

$$I_2 := \left| \mathbb{E} \int_{\tau(\bar{x}, t) \wedge T}^{\tau(x, t) \wedge T} f(X_s(\bar{x}, t)) \right| \leq \|f\|_{\kappa}^0 \mathbb{E} \int_{\tau(\bar{x}, t) \wedge T}^{\tau(x, t) \wedge T} e^{\kappa(T-s)} ds = \frac{1}{\kappa} \|f\|_{\kappa}^0 \mathbb{E} \left[e^{\kappa(T-T \wedge \tau(\bar{x}, t))} - e^{\kappa(T-T \wedge \tau(x, t))} \right]$$

and consequently, there exists $L > 0$ such that

$$e^{-\kappa(T-t)} I_2 \leq \frac{1}{\kappa} \|f\|_{\kappa}^0 \mathbb{E} \left[e^{\kappa(t-T \wedge \tau(\bar{x}, t))} - e^{\kappa(t-T \wedge \tau(x, t))} \right] \leq \frac{L}{\kappa} \|f\|_{\kappa}^0 |T \wedge \tau(\bar{x}, t) - T \wedge \tau(x, t)| \leq \frac{L}{\kappa} \|f\|_{\kappa}^0.$$

All inequalities can be now summarized into

$$e^{-\kappa(T-t)} |D_x u_f(x, t)| \leq \frac{L}{\kappa} \|f\|_{\kappa}^0,$$

which confirms that there exists a constant $M > 0$ such that

$$\|u\|_\kappa \leq \frac{M}{\kappa} \|f\|_\kappa^0.$$

□

We can define the mapping

$$(2.5) \quad \mathcal{T}u(x, t) := \mathbb{E}_{x,t} \left[\beta(X_{T \wedge \tau(x,t)}, T \wedge \tau(x, t)) + \int_t^{T \wedge \tau(x,t)} H(D_x u(X_s, s), u(X_s, s), X_s, s) ds \right].$$

Proposition 2.3. *If all conditions of Assumption 1 be satisfied, then the operator \mathcal{T} maps $C_{b,h}^{1,0}$ into $C_{b,h}^{1,0}$ and there exists a constant $\kappa > 0$ that the mapping (2.5) is a contraction in the norm $\|u\|_\kappa$.*

Proof. We have to prove first, that the operator \mathcal{T} maps $C_{b,h}^{1,0}$ into $C_{b,h}^{1,0}$. We fix the function $u \in C_{b,h}^{1,0}$ and define

$$w(x, t) = \mathbb{E} \left[\int_0^{T \wedge \tau(x,t)} H(D_x u(X_s, s), u(X_s, s), X_s, s) ds + \beta(X_{T \wedge \tau(x,t)}(x, t), T \wedge \tau(x, t)) \right].$$

The function w is bounded since u , $D_x u$, β are bounded and the Hamiltonian H satisfies the linear growth condition. The function β is uniformly Lipschitz continuous, which implies that there exists a constant $K > 0$ such that

$$\begin{aligned} & \mathbb{E} \left| \beta(X_{T \wedge \tau(x,t)}(x, t), T \wedge \tau(x, t)) - \beta(X_{T \wedge \tau(\bar{x},t)}(\bar{x}, t), T \wedge \tau(\bar{x}, t)) \right| \\ & \leq K \mathbb{E} \left| X_{T \wedge \tau(x,t)}(x, t) - X_{T \wedge \tau(\bar{x},t)}(\bar{x}, t) \right| + K \mathbb{E} |T \wedge \tau(x, t) - T \wedge \tau(\bar{x}, t)|. \end{aligned}$$

Moreover, since $X_{T \wedge \tau(x,t)}(x, t) > X_{T \wedge \tau(\bar{x},t)}(\bar{x}, t)$, we have

$$\mathbb{E} \left| X_{T \wedge \tau(x,t)}(x, t) - X_{T \wedge \tau(\bar{x},t)}(\bar{x}, t) \right| = \mathbb{E} X_{T \wedge \tau(x,t)}(x, t) - \mathbb{E} X_{T \wedge \tau(\bar{x},t)}(\bar{x}, t) = x - \bar{x}.$$

It is now easy to notice that the function w is Lipschitz continuous in x uniformly with respect to t , and consequently the function $D_x w$ is bounded. Moreover, w is a classical solution to parabolic differential equation

$$\begin{cases} D_t + \frac{1}{2}a(x, t)D_x^2 u + f(x, t) = 0, & (x, t) \in (0, +\infty) \times [0, T), \\ u(x, t) = \beta(x, t), & (x, t) \in \partial((0, +\infty) \times [0, T)), \end{cases}$$

which guarantees that $D_x w$ is Hölder continuous on compact subsets and consequently \mathcal{T} maps $C_{b,h}^{1,0}$ into $C_{b,h}^{1,0}$.

Now our aim is to prove that \mathcal{T} is a contraction for sufficiently large κ . Let's fix $u, v \in C_b^{1,0}$ and define

$$\bar{w}(x, t) = \mathcal{T}u(x, t) - \mathcal{T}v(x, t).$$

Note that \bar{w} is a classical solution to

$$(2.6) \quad \begin{cases} w_t + \frac{1}{2}a(x, t)D_x^2 w + H(D_x u, u, x, t) - H(D_x v, v, x, t) = 0, & (x, t) \in \mathbb{R}^N \times [0, T), \\ u(x, t) = 0, & (x, t) \in \partial((0, +\infty) \times [0, T)). \end{cases}$$

After applying Proposition 2.2 we get that there exists a constant $M > 0$, that

$$\|w\|_\kappa \leq \frac{M}{\kappa} \|H(D_x u, u, x, t) - H(D_x v, v, x, t)\|_\kappa^0 \leq \frac{M}{\kappa} \|u - v\|_\kappa.$$

This completes the proof. \square

Theorem 2.4. *Assume that all conditions from Assumption 1 are satisfied. Then there exist a classical solution to (2.1), which belongs to the class $u \in \mathcal{C}^{2,1}((0, +\infty) \times [0, T)) \cap \mathcal{C}([0, +\infty) \times [0, T])$ and in addition is bounded together with $D_x u$.*

Proof. The proof is analogous to the proof of [9, Theorem 2.2], but we repeat it for the reader's convenience. The reasoning is based on a fixed point type argument for the mapping \mathcal{T} . We take any $u_1 \in C_{b,h}^{1,0}$ and define recursively the sequence

$$u_{n+1} = \mathcal{T}u_n, \quad n \in \mathbb{N}.$$

There exists $\kappa > 0$ such that the mapping \mathcal{T} is a contraction in $\|\cdot\|$ and this implies that the sequence u_n is convergent to some fixed point u . But u belongs to $C_b^{1,0}$ and we have to prove that u belongs also to the class $C_{b,h}^{1,0}$. Let us note first that functions u_n and $D_x u_n$ are convergent in $\|\cdot\|_\kappa$ (for κ large enough), thus they are bounded uniformly with respect to n . We can now exploits (E8), (E9), (E10) from Fleming and Rishel [5] and prove uniform bound on compact subsets for Hölder norm of $D_x u_n$. Therefore $D_x u$ is Hölder continuous in x on compact subsets uniformly with respect to t . This confirms that the fixed point u belongs to the class $\mathcal{C}^{2,1}((0, +\infty) \times [0, T)) \cap \mathcal{C}([0, +\infty) \times [0, T])$ and satisfies equation (2.1). \square

Proposition 2.5. *Let the function $\sigma : (0, +\infty) \rightarrow \mathbb{R}$ be bounded, bounded away from zero and uniformly Lipschitz continuous. Then condition (2.3) is satisfied for the hitting time $\tau(x, t)$ of the following SDE*

$$dX_t = \sigma(X_t)dW_t.$$

Proof. The proof consists of three parts.

Step 1 First, we consider trivial dynamics of the form

$$dX_t = dW_t$$

Note that

$$\begin{aligned}
 (2.7) \quad \mathbb{E}\tau(x, t) \wedge T &= \int_t^T P(\tau(x, t) > s) ds = \int_t^T P(\tau(x, t) > s) ds = \int_t^T [1 - P(\tau(x, t) \leq s)] ds \\
 &= \int_t^T [1 - 2P(W_s - W_t > -x)] ds.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 (2.8) \quad |\mathbb{E}\tau(x, t) \wedge T - \mathbb{E}\tau(\bar{x}, t) \wedge T| &= \int_t^T \int_x^y \frac{1}{2\pi\sqrt{s-t}} e^{-\frac{z^2}{2(s-t)}} dz ds \\
 &\leq |x - \bar{x}| \int_t^T \frac{1}{2\pi\sqrt{s-t}} ds \leq \frac{\sqrt{T}}{\sqrt{\pi}} |x - \bar{x}|.
 \end{aligned}$$

Step 2 In the next step we consider dynamics of the form

$$(2.9) \quad dX_t = b(X_t)dt + dW_t,$$

where the function b is bounded and Lipschitz continuous on compact subsets of $[0, +\infty)$. In Proposition 2.4 we proved that the equation

$$\begin{cases} u_t + \frac{1}{2}D_x^2 u + b(x)D_x u - 1 = 0, & (x, t) \in \mathbb{R}^N \times [0, T), \\ u(x, t) = 0 & \partial((0, +\infty) \times [0, T)) \end{cases}$$

admits a classical solution u with bounded derivative $D_x u$. The standard verification theorem ensures that $u(x, t) = \mathbb{E}T \wedge \tau(x, t) - t$. So the condition (2.3) is satisfied for (2.9) as well.

Suppose now, that $\sigma \in C_{b,loc}^{1+1}$ and consider the dynamics

$$dY_t = -\frac{1}{2} \frac{\sigma_x(\zeta(Y_t))}{\sigma(\zeta(Y_t))} + dW_t.$$

We need as well the function

$$\zeta(x) = \int_0^x \sigma(z) dz,$$

which belongs to the class \mathcal{C}^2 . By the Itô, formula we get that $X_t = \zeta(Y_t)$ is the unique strong solution to

$$dX_t = \sigma(X_t) dW_t.$$

We have

$$\{k > 0 | Y_k(\zeta^{-1}(x), t) = 0\} = \{k > 0 | X_k(x, t) = 0\}.$$

Condition ((2.3)) is satisfied for the process Y_t and using the fact that ζ^{-1} is a Lipschitz continuous function we get the same for the process X .

Step 3 In the third step we consider the sequence of mollifiers $(\zeta_n | n \in \mathbb{N})$ and define the sequence

$$\sigma_n(x) = \zeta_n \star \sigma(x), \quad n \in \mathbb{N},$$

and the sequence of diffusions

$$dX_t^n = \sigma_n(X_t^n, t)dt,$$

and finally the sequence of stopping times

$$\tau_n(x, t) = \{k > 0 \mid X_k^n(x, t) = 0\}.$$

We deduce from the proof of Theorem 2.4 that it is constant $K' > 0$ such that for all $n \in \mathbb{N}$

$$|\mathbb{E}T \wedge \tau_n(x, t) - \mathbb{E}T \wedge \tau_n(\bar{x}, t)| \leq K'|x - \bar{x}|,$$

where the constant K' is independent of n . Passing to the limit, we get

$$|\mathbb{E}T \wedge \tau(x, t) - \mathbb{E}T \wedge \tau(\bar{x}, t)| \leq K'|x - \bar{x}|.$$

□

3. STOCHASTIC CONTROL APPLICATIONS

Here we adapt our result to be applicable for stochastic control problems. We consider the HJB equation of the form

$$(3.1) \quad D_t + \frac{1}{2}\sigma^2(x, t)D_x^2 u + \max_{\delta \in D} (i(x, t, \delta)D_x u + h(x, t, \delta)u + f(x, t, \delta)) = 0,$$

$$(x, t) \in (0, +\infty) \times [0, T]$$

with the boundary condition $u(x, t) = \beta(x, t)$, for $(x, t) \in \partial((0, +\infty) \times [0, T])$.

Assumption 2.

- B1)** *The coefficient $\sigma > \varepsilon > 0$ is bounded, Lipschitz continuous on compact subsets in $[0, +\infty) \times [0, T]$, and Lipschitz continuous in x uniformly with respect to t .*
- B2)** *Functions f, h, i are continuous and bounded and there exists a constant $L > 0$ such that for all $\zeta = f, h, i$ and for all $\delta \in D$, $(x, t) \in [0, +\infty) \times [0, T]$*

$$|\zeta(x, t, \delta) - \zeta(\bar{x}, \bar{t}, \delta)| \leq L(|x - \bar{x}| + |t - \bar{t}|).$$

- B3)** *The function β is Lipschitz continuous.*

- B4)** *There exists a constant $L > 0$ that for all $(x, t), (\bar{x}, t) \in \mathbb{R} \times [0, T]$*

$$|\mathbb{E}\tau(x, t) \wedge T - \mathbb{E}\tau(\bar{x}, t) \wedge T| \leq L|x - \bar{x}|.$$

Now we can give the immediate consequence of the Theorem 2.4.

Proposition 3.1. *Assume that all conditions of Assumption 2 are satisfied. Then there exists smooth solution to the problem (3.1)*

Optimal restricted dividend problem We consider an insurance company and its surplus of the form:

$$dX_k = [g(X_k) - c_k]dt + \sigma dW_k, \quad X_t = x, \quad t \leq k \leq T,$$

where the process $(c_t, 0 \leq t \leq T)$ denotes the stream of dividends. In the literature we can find variety of problems of the form:

$$J^c(x, t) = \mathbb{E}_{x,t} \left[\int_t^{T \wedge \tau(x,t)} (e^{-rk} U(c_k, X_k) + e^{-rT \wedge \tau(x,t)} \beta(X_{T \wedge \tau(x,t)})) dk \right].$$

The function U we can interpret as the utility function and $r > 0$ is the discount rate.

The insurance company wants to maximize $J^c(x, t)$ over the set of progressively measurable processes $(c_t, 0 \leq t \leq T)$ taking values in a fixed compact set $[m_1, m_2]$. Here, we can use the HJB of the form:

$$D_t + \frac{1}{2} \sigma^2 D_x^2 u + \max_{m_1 \leq c \leq m_2} [[g(x) - c] D_x u + U(c, x)] - ru = 0$$

For the discussion about recent advances of theory of dividend problems see Avanzi [1] or Zhu [10].

4. THE OPTIMAL CONSUMPTION PROBLEM WITH UNCERTAIN HORIZON

Our investor has an access to two securities: a bank account $(B_t, 0 \leq t < +\infty)$ and a share $(S_t, 0 \leq t < +\infty)$. We assume also that the price of the share depends on one non-tradable (but observable) factor $(Y_t, 0 \leq t < +\infty)$. This factor can represent an additional source of an uncertainty, here we can assume that this process will determine the investment horizon. Namely, let us define

$$\tau(y, t) =: \inf\{s > t : Y_s(y, t) = y_0\}.$$

Processes mentioned above are solutions to the system of stochastic differential equations

$$(4.1) \quad \begin{cases} dB_t &= r(Y_t) B_t dt, \\ dS_t &= [r(Y_t) + b(Y_t)] S_t dt + \sigma(Y_t) S_t dW_t^1, \\ dY_t &= g(Y_t) dt + a(Y_t) (\rho dW_t^1 + \bar{\rho} dW_t^2). \end{cases}$$

The dynamics of the investors wealth process $(X_t^{\pi, c}, 0 \leq t \leq T)$ is given by the stochastic differential equation

$$(4.2) \quad \begin{cases} dX_k = [r(Y_k) X_k + \pi_k b(Y_k) X_k] dk + \pi_k \sigma(Y_k) X_k dW_k^1 - c_k X_k dk, \\ X_t = x, \end{cases}$$

where x denotes the current wealth of the investor, $(\pi_k, t \leq k \leq T)$ is part of the wealth invested in S_t , $(c_k, t \leq k \leq T)$ is the consumption intensity process. The objective for the

investor looks as follows

$$J^{\pi,c}(x, y, t) = \mathbb{E}_{x,t} \int_t^{T \wedge \tau(y,t)} e^{-wk} U(c_k X_k) dk + e^{-wT \wedge \tau(y,t)} U(X_{T \wedge \tau(y,t)}),$$

where $U(x) = \frac{x^\gamma}{\gamma}$. The investor's aim is to maximize $J^{\pi,c}(x, y, t)$ with respect to $(\pi, c) \in \mathcal{A}$, which is not described here in detail.

To solve it we use a HJB equation of the form

$$(4.3) \quad V_t + \frac{1}{2}a^2 V_{yy} + \max_{\pi \in \mathbb{R}} \left(\frac{1}{2}\pi^2 \sigma^2(y) x^2 V_{xx} + \rho\pi\sigma(y)a(y)xV_{xy} + \pi b(y)xV_x \right) + g(y)V_y + \max_{c>0} (-cxV_x + c^\gamma x^\gamma) + (r(y) - w)V = 0,$$

with boundary conditions $V(x, y, T) = \frac{1}{\gamma}x^\gamma$, $V(x, y_0, t) = \frac{1}{\gamma}x^\gamma$.

Calculating both maxima and plugging into the equation we get

$$(4.4) \quad F_t + \frac{1}{2}a^2(y)F_{yy} + \frac{\rho^2\gamma}{2(1-\gamma)}a^2(y)\frac{F_y^2}{F} + \left(g(y) + \frac{\rho\gamma}{1-\gamma}a(y)\lambda(y) \right) F_y + \frac{\gamma}{2(1-\gamma)}\lambda^2(y)F + \gamma r(y)F + (1-\gamma)F^{\frac{\gamma}{\gamma-1}} - wF = 0,$$

with boundary conditions $F(y, T) = 1$, $F(y_0, t) = 1$. Moreover, the optimal portfolio/consumption candidate is given by

$$\pi^*(x, y, t) = \frac{\rho a(y)}{(1-\gamma)\sigma(y)} \frac{F_y}{F} + \frac{\lambda(y)}{(1-\gamma)\sigma(y)},$$

$$c^*(x, y, t) = F^{\frac{1}{\gamma-1}}.$$

To simplify the equation we follow Zariphopoulou [8] and use the transformation

$$F(y, t) = G^\delta(y, t), \quad \delta = \frac{1-\gamma}{\gamma\rho^2 + 1 - \gamma}.$$

This will reduce the equation to the form

$$(4.5) \quad G_t + \frac{1}{2}a^2(y)G_{yy} + \left(g(y) + \frac{\gamma\rho}{1-\gamma}a(y)\lambda(y) \right) G_y + \left(\frac{\gamma}{2\delta(1-\gamma)}\lambda^2(y)G \right) + \frac{\gamma}{\delta}r(y)G + \frac{1-\gamma}{\delta}G^{1-\frac{\delta}{1-\gamma}} = 0.$$

Note, that $0 < 1 - \frac{\delta}{1-\gamma} < 1$ and there exists $\alpha \in (0, 1)$ that

$$\frac{\alpha}{\alpha-1} = 1 - \frac{\delta}{1-\gamma}.$$

and consequently

$$G^{1-\frac{\delta}{1-\gamma}} = \max_{c>0} (-\alpha c G + c^\alpha)$$

Therefore, it is reasonable to consider first HJB equations of the form

$$(4.6) \quad G_t + \frac{1}{2}a^2(y)G_{yy} + i(y)G_y + h(y)G + \max_{m_1 \leq c \leq m_2} (-\theta\alpha c G + \theta c^\alpha) = 0, \quad \theta > 0.$$

with the boundary condition $G(y, t) = 1$, for $(y, t) \in \partial((y_0, +\infty) \times [0, T))$.

Proposition 4.1. *Suppose that functions σ , h , i are Lipschitz continuous and bounded and in addition let σ be bounded away from zero. Then there exists $G \in \mathcal{C}^2((0, +\infty) \times [0, T]) \cap \mathcal{C}([0, +\infty) \times [0, T])$ which satisfies (4.5) and is bounded together with its first derivative with respect to y .*

Proof. Thanks to our theorem we know that equation (4.6) has a bounded classical solution which is bounded together with the first derivative G_y .

First we need to obtain uniform bounds for the function G . The standard verification theorem guarantees that

$$G_{m_1, m_2}(y, t) = \sup_{c \in \mathcal{C}_{m_1, m_2}} \mathbb{E}_{y, t} \left[\int_t^{T \wedge \tau(y, t)} e^{\int_t^s (h(Y_k) - \theta \alpha c_k) dk} c_s^\alpha ds + e^{\int_t^{T \wedge \tau(y, t)} (h(Y_k) - \theta \alpha c_k) dk} \right].$$

Since the function h is bounded, there exists a constant $D > 0$ such that for all $(y, t) \in [0, +\infty) \times [0, T]$

$$\begin{aligned} |G_{m_1, m_2}(y, t)| &\leq \sup_{c \in \mathcal{C}_{m_1, m_2}} \mathbb{E}_{y, t} \left(\int_t^{T \wedge \tau(y, t)} e^{\int_t^s (h(Y_k) - \theta \alpha c_k) dk} c_s^\alpha ds \right) \\ &\leq D \sup_{c \in \mathcal{C}_{m_1, m_2}} \mathbb{E}_{y, t} \left[\int_t^T e^{-\int_t^s \theta \alpha c_k dk} c_s^\alpha ds + 1 \right]. \end{aligned}$$

Furthermore, note that

$$\int_t^T e^{-\int_t^s \theta \alpha c_k dk} c_s^\alpha ds \leq \int_t^T e^{-\int_t^s \theta \alpha c_k dk} \chi_{\{c_s \leq 1\}} ds + \int_t^T e^{-\int_t^s \theta \alpha c_k dk} c_s \chi_{\{c_s > 1\}} ds$$

and

$$\int_t^T e^{-\int_t^s \theta \alpha c_k dk} c_s ds = \frac{1}{\theta \alpha} \left[-e^{-\int_t^s \theta \alpha c_k dk} \right]_t^T = \frac{1}{\theta \alpha} \left[1 - e^{-\int_t^T \theta \alpha c_k dk} \right].$$

Thus,

$$|G_{m_1, m_2}(y, t)| \leq D_1(T + 1).$$

Inserting $c \equiv 1$ and $q \equiv 0$, we get

$$\begin{aligned} G_{m_1, m_2}(y) &= \sup_{c \in \mathcal{C}_{m_1, m_2}} \mathbb{E}_{y, t} \left[\theta \int_t^{\tau(y, t) \wedge T} e^{\int_t^s (h(Y_k) - \theta \alpha c_k) dk} c_s^{\theta \alpha} ds + e^{\int_t^{T \wedge \tau(y, t)} (h(Y_k) - \theta \alpha c_k) dk} \right] \\ &\geq \mathbb{E}_{y, t} \left[e^{\int_t^{T \wedge \tau(y, t)} (h(Y_k) - \theta \alpha) dk} \right] \geq e^{KT}, \end{aligned}$$

where $K < 0$ is any constant such that $K \leq \inf_y (h(y) - \theta \alpha)$.

Thus, there exist $p, P > 0$ such that

$$p \leq G_{m_1, m_2}(y) \leq P, \quad m_1, m_2 \in (0, +\infty).$$

This ensures that there exist a pair m_1^*, m_2^* such that

$$m_1^* \leq G_{m_1^*, m_2^*}^{1 - \frac{\delta}{1-\gamma}} \leq m_2^*,$$

$$G_{m_1^*, m_2^*}^{1 - \frac{\delta}{1-\gamma}} = \max_{c > 0} \left(-\theta \alpha c G_{m_1^*, m_2^*} + \theta c^\alpha \right) = \max_{m_1^*(n) \leq c \leq m_2^*} \left(-\theta \alpha c G_{m_1^*, m_2^*} + \theta c^\alpha \right),$$

and finally $G := G_{m_1^*, m_2^*}$ is a solution to (4.5). The boundedness condition for $D_y G$ is proved by finding the uniform bound for the Lipschitz constant. We have

$$\begin{aligned} |G(y, t) - G(\bar{y}, t)| &\leq \sup_{c \in \mathcal{C}_{m_1^*, m_2^*}} \left| \mathbb{E} \int_t^{T \wedge \tau(y, t)} \left[e^{\int_t^s h(Y_k(y, t)) dk} - e^{\int_t^s h(Y_k(\bar{y}, t)) dk} \right] e^{-\int_t^s \theta \alpha c_k dk} c_s^\alpha ds \right| \\ &\quad + \sup_{c \in \mathcal{C}_{m_1^*, m_2^*}} \left| \mathbb{E} \left[e^{\int_t^{T \wedge \tau(y, t)} (h(Y_k(y, t)) - \theta \alpha c_k) dk} - e^{\int_t^{T \wedge \tau(y, t)} (h(Y_k(\bar{y}, t)) - \theta \alpha c_k) dk} \right] \right| \\ &\quad + \sup_{c \in \mathcal{C}_{m_1^*, m_2^*}} \left| e^{\int_t^{T \wedge \tau(y, t)} (h(Y_k(\bar{y}, t)) - \theta \alpha c_k) dk} - e^{\int_t^{T \wedge \tau(\bar{y}, t)} (h(Y_k(\bar{y}, t)) - \theta \alpha c_k) dk} \right| \\ &\quad + \sup_{c \in \mathcal{C}_{m_1^*, m_2^*}} \left| \mathbb{E} \int_{T \wedge \tau(\bar{y}, t)}^{T \wedge \tau(y, t)} e^{\int_t^s h(Y_k(\bar{y}, t)) dk} e^{-\int_t^s \theta \alpha c_k dk} c_s^\alpha ds \right| \end{aligned}$$

The function h is Lipschitz continuous in y and this implies that

$$|h(Y_k(y, t)) - h(Y_k(\bar{y}, t))| \leq L_1 |Y_k(y, t) - Y_k(\bar{y}, t)| \leq \sup_{t \leq k \leq T} |Y_k(y, t) - Y_k(\bar{y}, t)|.$$

Thus,

$$\left| e^{\int_t^s h(Y_k(\bar{y}, t)) dk} - e^{\int_t^s h(Y_k(y, t)) dk} \right| \leq L_1 e^{MT} \sup_{t \leq k \leq T} |Y_k(y, t) - Y_k(\bar{y}, t)|$$

and

$$\left| e^{\int_t^{T \wedge \tau(y, t)} h(Y_k(y, t)) dk} - e^{\int_t^{T \wedge \tau(y, t)} h(Y_k(\bar{y}, t)) dk} \right| \leq L_1 e^{MT} \sup_{t \leq k \leq T} |Y_k(y, t) - Y_k(\bar{y}, t)|.$$

Since $c_s \leq m_1^*$, we can notice that there exists a constant $M' > 0$ such that

$$|G(y, t) - G(\bar{y}, t)| \leq M' \left(\mathbb{E} \sup_{t \leq k \leq T} |Y_k(y, t) - Y_k(\bar{y}, t)| + |\mathbb{E} T \wedge \tau(y, t) - \mathbb{E} T \wedge \tau(\bar{y}, t)| \right).$$

The conclusion of Theorem 2.4 and standard estimates for stochastic differential equations ensure that $D_y G$ is uniformly bounded and this completes the proof. \square

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